Transient Analysis of Tapered Lines Based on the Method of Series Expansion

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***Abstract*— Step response of tapered lines under matched conditions is studied in detail by a novel method of series expansion. We use the concept of fall time to characterize the transient characteristics and, by using approximate formulas concerned with Gaussian pulse, response characteristics are derived which may be a guide in practical applications of tapered line. The effectiveness of the formulas is verified by numerical calculations.**

I. Introduction

Tapered or nonuniform transmission lines are widely used as broad band matching sections in microwave circuits, which explains the abundance of previous studies on tapered lines in the frequency domain [1], [2]. However, the transient characteristics of tapered lines are largely ignored, which is essential especially when the tapered line is applied as a pulse-matching section [3]–[5]. Recently, Hsue and Hechtman studied the transient behavior of general tapered lines by investigating the step response and concluded that the exponential line provides the largest first arriving wave followed by a decaying transient ripple and, thus, had potential application as pulse transformer [6], [7]. A revised conclusion is obtained in [8] by Tang *et al.* to the effect that the advantage of the exponential line lies in its minimum dropping speed instead of in its maximum first arriving wave.

In this paper, the step response of tapered lines under matched conditions is studied by a novel method of series expansion, which is a general way in studying the transient characteristics of tapered lines. We use the concept of fall time to characterize the transient characteristics of tapered lines, and it is shown that the performance of tapered lines used as pulse matching sections is largely determined by the ratio of the width of the exciting pulse to the fall time. Tapered lines work effectively only when the width of the exciting pulse is less than or comparable to its fall time. Approximate formulas, concerned with the response characteristics under Gaussian pulse excitation are derived here, which are verified by subsequent numerical calculations, and may be an effective guide in practical applications of tapered lines.

II. Theoretical Analysis of Step Response

A. Transform

With quasi-TEM mode approximation lossless transmission lines are governed by the following standard equations [8]:

$$\frac{\partial V}{\partial x_1} + L\frac{\partial I}{\partial t_1} = 0 \tag{1a}$$

$$C\frac{\partial V}{\partial t_1} + \frac{\partial I}{\partial x_1} = 0 \tag{1b}$$

where $C(x_1), L(x_1)$ are the capacitance, inductance per unit length, respectively. The propagation speed and characteristic impedance are, respectively, given by

$$P_S(x_1) = 1/\sqrt{L(x_1)C(x_1)}, \quad Z(x_1) = \sqrt{L(x_1)/C(x_1)}. \tag{2}$$

With no consideration of dispersion, we use the following transformations as used in [8]:

$$x = \frac{1}{T_d}\int_0^{x_1} \frac{dx_1}{P_S(x_1)}, \quad t = \frac{t_1}{T_d} \tag{3a}$$

$$p = \frac{V}{\sqrt{Z}} + I\sqrt{Z}, \quad q = \frac{V}{\sqrt{Z}} - I\sqrt{Z} \tag{3b}$$

where

$$T_d = \int_0^l \frac{dx_1}{P_S(x_1)} \tag{4}$$

is the propagation delay of the line. Then, equations (1a) and (1b) become

$$\frac{\partial q}{\partial t} - \frac{\partial q}{\partial x} = r(x)p \tag{5a}$$

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} = -r(x)q \tag{5b}$$

where $0 \le x \le 1, 0 \le t < \infty$ and

$$r(x) = \frac{dZ(x)}{dx}\frac{1}{2Z(x)} \tag{6}$$

for tapered lines, $r(x)$ is assumed continuous and derivable.

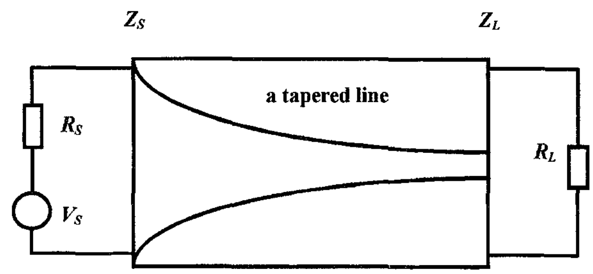

Fig. 1. A tapered line used as a matching section.

A tapered line as a matching section is illustrated in Fig. 1 where R_S and R_L denote the source and load impedances, Z_S and Z_L denote the characteristic impedances at the source and load ends, V_S is the source voltage and V_L is the voltage at the load end. Under well matched condition, $R_S = Z_S$ and $R_L = Z_L$ are satisfied and, with the transform of (3), the boundary conditions becomes

$$p(0,t) = \frac{V_S(t)}{\sqrt{Z_S}}, \quad q(1,t) = 0. \tag{7}$$

B. Method of Series Expansion

Assuming $R_S = Z_S = Z_0, R_L = Z_L = Z_1$ and V_S is a unit step source, by taking Laplace transform of both sides of (5a) and (5b) (with respect to t), we obtain

$$sq(x,s) - \frac{dq(x,s)}{dx} = r(x)p(x,s) \tag{8a}$$

$$sp(x,s) + \frac{dp(x,s)}{dx} = -r(x)q(x,s) \tag{8b}$$

while the boundary conditions are turned into

$$q(1,s) = 0$$
$$p(0,s) = \frac{1}{\sqrt{Z_0}s}. \tag{9}$$

Manuscript received October 17, 1995; revised June 14, 1996.

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Publisher Item Identifier S 0018-9480(96)06913-X.

Now, by considering the power series approximation (with respect to t) of $P(x,t)$ and $q(x,t)$ near $t = x$, we suppose

$$\begin{aligned} p(x,t) &= a_0(x) + a_1(x)(t-x) + a_2(x)(t-x)^2 \\ &\quad + \Delta_1(x, t-x) \quad \text{for} \quad t - x \geq 0 \\ p(x,t) &= 0 \quad \text{for} \quad f - x < 0 \\ q(x,t) &= b_0(x) + b_1(x)(t-x) + b_2(x)(t-x)^2 \\ &\quad + \Delta_2(x, t-x) \quad \text{for} \quad t - x \geq 0 \\ q(x,t) &= 0 \quad \text{for} \quad t - x < 0 \end{aligned} \tag{10}$$

where $\Delta_1 = o((t-x)^2), \Delta_2 = o((t-x)^2)$ as $t - x \to 0$. Then, by taking Laplace transform, under general conditions we can obtain

$$\begin{aligned} p(x,s) &= \left(\frac{a_0(x)}{s} + \frac{a_1(x)}{s^2} + \frac{a_2(x)}{s^3} + \Delta_1(x,s)\right)e^{-sx} \\ q(x,s) &= \left(\frac{b_0(x)}{s} + \frac{b_1(x)}{s^2} + \frac{b_2(x)}{s^3} + \Delta_2(x,s)\right)e^{-sx} \end{aligned} \tag{11}$$

where

$$\lim_{s\to\infty} s^3\Delta_1(x,s) = 0, \quad \lim_{s\to\infty} s^3\Delta_2(x,s) = 0. \tag{12}$$

By substituting (11) into (8a) and (8b), and using (12), we obtain

$$\begin{aligned} b_0(x) &= 0 \\ 2b_1(x) - \frac{db_0(x)}{dx} &= r(x)a_0(x) \\ 2b_2(x) - \frac{db_1(x)}{dx} &= r(x)a_1(x) \\ \frac{da_1(x)}{dx} &= -r(x)b_i(x) \qquad i = 0,1,2. \end{aligned} \tag{13}$$

Numerical calculations show that $a_0(x), a_1(x)$ and $a_2(x)$ are continuous at $x = 0$ and $x = 1$ while $b_1(x)$ and $b_2(x)$ are discontinuous at $x = 1$. Thus, (13) is valid only for $0 \leq x < 1$. Making use of (9), we also have

$$\begin{aligned} b_i(1) &= 0 \qquad i = 0,1,2 \\ a_0(0) &= \frac{1}{\sqrt{Z_0}} \\ a_i(0) &= 0 \qquad i = 1,2 \end{aligned} \tag{14}$$

By using both (13) and (14), we can obtain the following results

$$\begin{aligned} a_0(x) &= \frac{1}{\sqrt{Z_0}}, \\ a_1(x) &= -\frac{1}{2\sqrt{Z_0}}\int_0^x r^2(x)\,dx, \\ a_2(x) &= \frac{1}{8\sqrt{Z_0}}\left[\left(\int_0^x r^2(x)\,dx\right)^2 - r^2(x) + r^2(0)\right] \end{aligned} \tag{15}$$

By (3), we finally obtain the unit step response as

$$\begin{aligned} S_L(s) = &\left(\frac{a_0(1)\sqrt{Z_1}}{2s} + \frac{a_1(1)\sqrt{Z_1}}{2s^2} + \frac{a_2(1)\sqrt{Z_1}}{2s^3}\right. \\ &\left. + \frac{\sqrt{Z_1}}{2}\Delta_1(1,s)\right)e^{-s} \end{aligned} \tag{16}$$

where

$$\frac{a_0(1)\sqrt{Z_1}}{2} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_0}} \tag{17a}$$

is the magnitude of the first arriving wave and

$$\frac{a_1(1)\sqrt{Z_1}}{2} = -\frac{1}{2}\sqrt{\frac{Z_1}{Z_0}} \times \frac{1}{2}\int_0^1 r^2(x)\,dx \tag{17b}$$

is the instantaneous dropping speed at $t = 1$, while

$$\begin{aligned} \frac{a_2(1)\sqrt{Z_1}}{2} = \frac{1}{16}\sqrt{\frac{Z_1}{Z_0}} \times &\left[\left(\int_0^1 r^2(x)\,dx\right)^2 - r^2(1)\right. \\ &\left. + r^2(0)\right] \end{aligned} \tag{17c}$$

is the second forward derivative at $t = 1$.

The above results show that it is not the first arriving wave but the following dropping process that reflects the transient characteristics of different tapered lines, which is concordant with [8]. Therefore, we use the concept of fall time T_f to characterize the transient characteristics, which is defined as the duration for S_L to decrease to 90% of the first arriving voltage. Assuming the dropping is linear at $t = 1$, by (17b) and (3a), we approximately have

$$T_f = \frac{0.1T_d}{\frac{1}{2}\int_0^1 r^2(x)\,dx}. \tag{18}$$

Further theoretical and numerical results both show that, the behavior of a tapered line approaches the behavior of an ideal pulse transformer when its fall time (in response to a step excitation) is much larger than the width of the exciting pulse (with no consideration of the propagation delay) or, conversely, tapered lines work effectively only when the width of the exciting pulse is less than or comparable to the fall time associated with a step excitation.

III. Response Characteristics Under Gaussian Pulse Excitation

The general pulse response is given by

$$V_L(t_1) = \int_0^{t_1} p(\tau)h(t_1 - \tau)\,d\tau \tag{19}$$

where $p(t_1)$ is the exciting pulse and $h(t_1)$ is the unit impulse response (with no consideration of the propagation delay). By using (17a), we divide $h(t_1)$ into two parts as

$$h(t_1) = \frac{1}{2}\sqrt{\frac{Z_1}{Z_0}}[\delta(t_1) + f(t_1)] \tag{20}$$

where $\delta(t_1)$ is a unit impulse function and $f(t_1) = dS_L(t_1)/dt_1 (t_1 > 0)$. $S_L(t_1)$ is the unit step response. By (17b) and (3a) we have

$$f(0) = \frac{-\frac{1}{2}\int_0^1 r^2(x)\,dx}{T_d} = -\frac{0.1}{T_f}. \tag{21}$$

Then, $V_L(t_1)$ is accordingly divided into two parts

$$V_L(t_1) = \frac{1}{2}\sqrt{\frac{Z_1}{Z_0}}\left[p(t_1) + \int_0^{t_1} p(\tau)f(t_1 - \tau)\,d\tau\right]. \tag{22}$$

For t_1 is less than or comparable to T_f, we make the approximation that $f(t_1) = -0.1/T_f$. Then, $V_L(t_1)$ is approximate to

$$\begin{aligned} V_L(t_1) &= \frac{1}{2}\sqrt{\frac{Z_1}{Z_0}}\left[p(t_1) - \frac{0.1}{T_f}\int_0^{t_1} p(\tau)\,d\tau\right] \\ &= \frac{1}{2}\sqrt{\frac{Z_1}{Z_0}}\left[p(t_1) - \frac{0.1t_1}{T_f}p(\xi)\right] \end{aligned} \tag{23}$$

where $0 \leq \xi \leq t_1$. Thus it can be seen that, when the width of $p(t_1)$ is less than or comparable to T_f, $V_L(t_1)$ reaches its peak value nearly at the same time as $p(t_1)$. Also, the following undershooting reaches its maximum value (valley point) as $p(t_1)$ tends to 0. We denote the peak and valley points value of $V_L(t_1)$ as V_{peak} and V_{valley}, respectively.

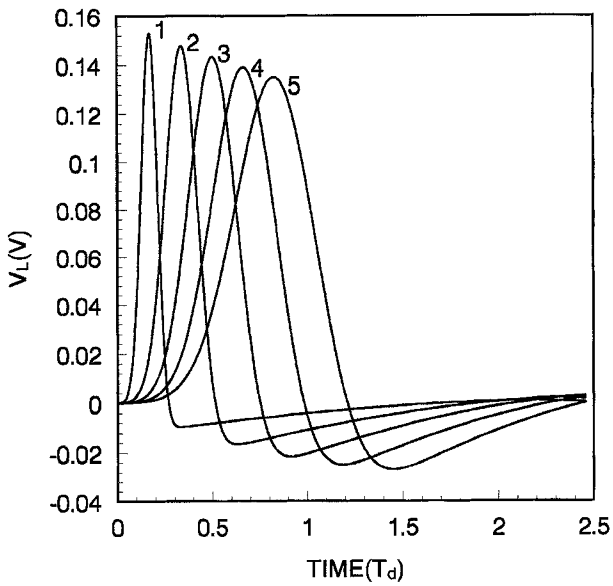

Fig. 2. The numerical Gaussian pulse responses for an exponential line, under the conditions $R_S = Z_S = 50\,\Omega, R_L = Z_L = 5\,\Omega$. 1: $W = 0.1T_d$, 2: $W = 0.2T_d$, 3: $W = 0.3T_d$, 4: $W = 0.4T_d$, 5: $W = 0.5T_d$.

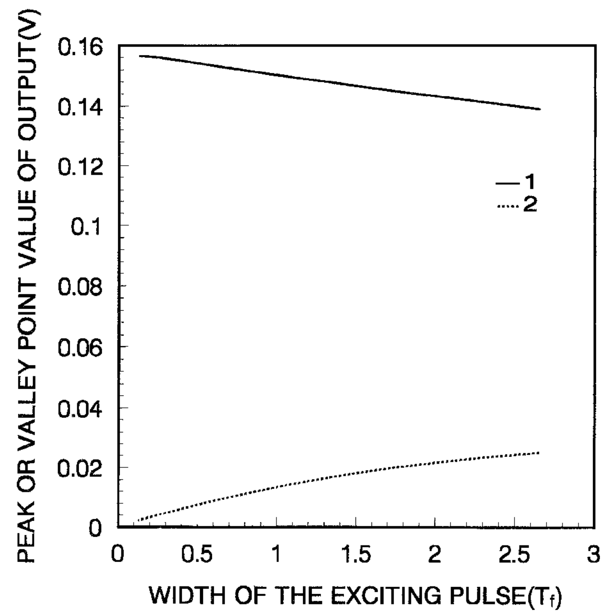

Fig. 3. Curves of the output peak value and of the valley point value of the following undershooting, as the width of the exciting pulse changes. 1: curve of the output peak value; 2: curve of the valley point value of the following undershooting.

For a Gaussian pulse excitation ($p(t_1) = A\exp(-(t_1 - T)^2/\sigma^2)$), we obtain the following approximate formulas

$$V_{\text{peak}} = V_i\left(1 - \frac{0.053W}{T_f}\right) \quad (24a)$$

$$V_{\text{valley}} = V_i\frac{0.106W}{T_f} \quad (24b)$$

where V_i is the peak value of $V_L(t_1)$ for an ideal transformer and W is the width of the exciting pulse at the half magnitude points.

IV. Numerical Results

In this section, numerical calculations are carried out to verify the preceding theoretical results. We solve (5a) and (5b) by the method of characteristics [8].

In Fig. 2, we show the numerical Gaussian pulse (with peak value = 1 V) responses for an exponential line under the conditions $R_S = Z_S = 50\,\Omega, R_L = Z_L = 5\,\Omega$. It can be seen that, the peak value of the output pulse decreases as the width of the excitation pulse increases, which implies that the tapered line become progressingly less effective as a matching section. Also, the following undershooting increases in its valley point value. In Fig. 3, for the same conditions that apply for Fig. 2, we show a curve of the output peak value and of the valley point value of the undershooting, as the width of the excitation Gaussian pulse changes from $0.1T_f$ to $2.6T_f$. For $1.5T_f \geq W \geq 0$, our numerical results are very near to the results calculated by (24a) and (24b). It is to be pointed out that, as the width of the exciting pulse increases, the error associated with (24b) increases more rapidly than that associated with (24a).

V. Conclusion

By the method of series expansion, we study the transient characteristics of dispersionless tapered lines more accurately as compared with [8]. By further deduction, we theoretically show that the coupling efficiency is determined by the ratio of the width of the exciting pulse to the fall time associated with a step impulse and that tapered lines work effectively only when the width of the exciting pulse is less than or comparable to the fall time for a step impulse, which may be an effective guide for practical applications.

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